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Expanding Hermitean Operators in a Basis of Projectors on Coherent Spin States

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Abstract

The expectation values of a hermitean operator $\widehat{A}$ in $(2s+1)^2$ specific coherent states of a spin are known to determine the operator unambiguously. As shown here, (almost) any other $(2s+1)^2$ coherent states also provide a basis for self-adjoint operators. This is proven by considering the determinant of the Gram matrix associated with the coherent state projectors as a Hamiltonian of a fictitious *classical* spin system.

State reconstruction [1] aims at parametrizing the density matrix $\hat{\rho}$ of a quantum system by the expectations of appropriately chosen observables, the quorum. For a spin s, the (unnormalized) density matrix has $N_s = (2s+1)^2$ independent real parameters; in [2], a particularly simple and non-redundant quorum consisting of precisely N_s projectors on coherent spin states $|\mathbf{n}\rangle$, satisfying $\mathbf{n} \cdot \widehat{\mathbf{S}}|\mathbf{n}\rangle = \hbar s|\mathbf{n}\rangle$, has been identified.

Indeed, the density matrix $\hat{\rho}$ of a spin s is determined unambiguously if one performs appropriate measurements with a traditional Stern-Gerlach apparatus. Distribute N_s axes $\mathbf{n}_n, n = 1, \ldots, N_s$, over $(2s+1)$ cones about the z axis with different opening angles in such a way that the set of the $(2s+1)$ directions on each cone is invariant under a rotation about z by an angle $2\pi/(2s+1)$. Then, an (unnormalized) statistical operator $\hat{\rho}$ is fixed by measuring the $(2s+1)^2$ relative frequencies $p_s(\mathbf{n}_n) = \langle \mathbf{n}_n|\hat{\rho}|\mathbf{n}_n\rangle$, that is, by the expectation values of the statistical operator $\hat{\rho}$ in the coherent states $|\mathbf{n}_n\rangle$. In other words, a hermitean operator $\widehat{A} \in \mathcal{A}_s$ (which is the space of linear operators acting in the Hilbert space $\mathcal{H}_s$ of the spin) is fixed by the values of its Q-symbol, $Q_A(\mathbf{n}) = \mathrm{Tr}[\widehat{A}|\mathbf{n}\rangle\langle\mathbf{n}|] = \langle\mathbf{n}|\widehat{A}|\mathbf{n}\rangle$ at N_s appropriately chosen points. For brevity, let us denote a set of N_s points (as well as the associated family of N_s unit vectors $\mathbf{n}_n$) as a 'constellation' $\mathcal{N}$ or a 'hedgehog' $\mathcal{N}$ with unit spikes $\mathbf{n}_n$. Independent reconstruction schemes for spin s do exist [3, 4].

For technical reasons, the spatial directions $\mathbf{n}_n$ dealt with in [2] were restricted to a certain class of *regular* hedgehogs, $\mathcal{N}_0$. The purpose here is to show that this restriction is not necessary: given a *generic* constellation $\mathcal{M}$, the N_s values of the Q-symbol $Q_A(\mathbf{n}_n)$ contain all the information about the operator $\widehat{A}$. Let us put it differently: given *any* constellation $\mathcal{M}$ of vectors $\mathbf{m}_n$, then *either* the numbers $Q_A(\mathbf{m}_n)$ determine $\widehat{A}$, *or* there is an *infinitesimally close* constellation $\mathcal{M}'$ such that the numbers $Q_A(\mathbf{m}'_n)$ do the job. Two hedgehogs $\mathcal{M}'$ and $\mathcal{M}$ are close if, for example, the number

$$d(\mathcal{M}', \mathcal{M}) = \sum_{n=1}^{N_s} |\mathbf{m}'_n - \mathbf{m}_n| \, , \tag{1}$$

is small. To visualise this statement, consider the real vector space $\mathbb{R}^3$: any three unit vectors form a basis provided they are neither co-planar nor co-linear. Among all possibilites, the exceptional constellations have measure zero. At the same time, it is obvious that arbitrarily small variations typically turn the three linearly dependent vectors into a basis of $\mathbb{R}^3$.

The starting point of the proof are N_s projection operators on coherent states,

$$\widehat{Q}_n = |\mathbf{n}_n\rangle\langle\mathbf{n}_n| \, , \quad \mathbf{n}_n \in \mathcal{N}^0 \, , \qquad 1 \leq n \leq N_s \, , \tag{2}$$

determined uniquely by the constellation $\mathcal{N}_0$ described. It will be shown now any other hedgehog $\mathcal{M}$ (or an infinitesimally close one, $\mathcal{M}'$) also will provide a basis of the space $\mathcal{A}_s$.

The N_s^2 elements of the *Gram matrix* $\mathsf{G}_{nn'}$ [5] associated with a constellation $\mathcal{M}$ are given by the scalar product of the projectors on coherent states:

$$\mathsf{G}_{nn'} = \text{ Tr } \left[\widehat{Q}_n \widehat{Q}_{n'}\right] = |\langle\mathbf{m}_n|\mathbf{m}_{n'}\rangle|^2 = \left(\frac{1 + \mathbf{m}_n \cdot \mathbf{m}_{n'}}{2}\right)^{2s} \, , \qquad 1 \leq n, n' \leq N_s \, . \tag{3}$$

Thus, the scalar product of two coherent states is a *polynomial* in the components of the associated unit vectors $\mathbf{m}_n$ and $\mathbf{m}_{n'}$. The result in [2] comes down to saying that the Gram matrix of the constellation $\mathcal{N}_0$ is invertible or, equivalently, its determinant does not vanish.

The determinant of the matrix G, if conceived as a function of the n-th vector, is infinitely often differentiable with respect to its components, according to (3). Upon keeping the vectors $\mathbf{n}_1, \ldots, \mathbf{n}_{n-1}$ and $\mathbf{n}_{n+1}, \ldots, \mathbf{n}_{N_s}$ fixed, it may be regarded as a fictitious time-independent *Hamiltonian function* H of a single classical spin, $\mathbf{n}_n$:

$$\det \mathsf{G}(\mathbf{n}_n) = H(\mathbf{n}_n) \, . \tag{4}$$

It is different from zero if $\mathbf{n}_n$ coincides with the n-th vector of the constellation $\mathcal{N}_0$. This Hamiltonian describes an *integrable* system since there is just one degree of freedom accompanied by one constant of motion, the Hamiltonian itself [6]. The two-dimensional phase space S^2 is foliated entirely by one-dimensional tori of constant energy. In addition, a finite number of (elliptic or hyperbolic) fixed points and one-dimensional separatrices will occur. This can be seen, for example, by looking at the flow on the unit sphere generated by the Hamiltonian $H(\mathbf{n}_n)$:

$$\frac{d\mathbf{n}_n}{dt} = \mathbf{n}_n \times \frac{\partial H}{\partial \mathbf{n}_n}, \tag{5}$$

where $\partial/\partial\mathbf{n}_n$ is the gradient with respect to $\mathbf{n}_n$ [7]. The right-hand-side is a (non-zero) polynomial in the components of $\mathbf{n}_n$, implying that the integral curves of the Hamiltonian are fixed points, separatrices, and closed orbits. This means that $H(\mathbf{n}_n)$ can take the value zero at a finite number of (open or closed) curves or points at most. Consequently, the determinant of $\mathsf{G}(\mathbf{n}_n)$ is different from zero for almost all choices of $\mathbf{n}_n$. Therefore, one can move the vector $\mathbf{n}_n$ into any other vector, including $\mathbf{m}_n$, the n-th vector of the desired constellation $\mathcal{M}$, thereby passing possibly through points with $\det \mathsf{G} = 0$. If, accidentally, $\mathbf{m}_n$ corresponds to a point with vanishing energy (this happens with probability zero only), one can nevertheless approach it arbitrarily close by a vector $\mathbf{m}'_n$ with $|\mathbf{m}'_n - \mathbf{m}_n| < \varepsilon/N_s$ since levels of constant energy have a co-dimension at most equal to one.

Working one's way from $n = 1$ to N_s, one ends up with a constellation $\mathcal{M}'$ which is guaranteed to be infinitesimally close to $\mathcal{M}$ since $\sum_n |\mathbf{m}'_n - \mathbf{m}_n| < \varepsilon$ can be made arbitrarily small. With probability one, the constellation $\mathcal{M}$ is obtained even exactly. Consequently, almost all hedgehogs $\mathcal{M}$ of N_s projection operators $\widehat{Q}_n$ give rise to a *basis* in the space of linear operators on $\mathcal{H}_s$, the Hilbert space of a spin s. In turn, the values of the *discrete* Q-symbol related to a constellation $\mathcal{M}$ are indeed sufficient to determine the operator $\widehat{A}$.

In summary, it has been shown that (almost) any distribution of N_s points on the sphere S^2 gives rise to a non-orthogonal basis of coherent-state projectors $\widehat{Q}_n$ in the linear space $\mathcal{A}_s$ of operators for a spin s. An independent proof of this result can be found in [8]. In addition, a discrete variant of the P-symbol is shown there to come along naturally with the discrete Q-symbol. The relation of the basis of projectors $\widehat{Q}_n$ to a symbolic calculus *à la* Stratonovich-Weyl has been elaborated in [9].

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